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Diferencialne operacije vektorske analize

Študijsko gradivo

Če ni drugače rečeno, so v nadaljevanju vsi vektorji izraženi v standardni bazi $\{\vec{i}, \vec{j}, \vec{k}\}$.

Krajevni vektor točke $T(x, y, z)$ je $\vec{r} = (x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$. V nadaljevanju pomenijo

$$\begin{aligned}\alpha, \beta, \dots & \text{ realne konstante,} \\ \mathcal{U}(\vec{r}), \mathcal{V}(\vec{r}), \dots & \text{ skalarna polja,} \\ \vec{F}(\vec{r}), \vec{G}(\vec{r}), \dots & \text{ vektorska polja.}\end{aligned}$$

Osnovni diferencialni operator prvega reda

Hamiltonov operator, nabra:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

Gradient skalarne polja \mathcal{U} :

$$\text{grad}\mathcal{U} = \vec{\nabla}\mathcal{U}.$$

Odvod skalarne polja \mathcal{U} v smeri enotskega vektorja \vec{e} :

$$\frac{\partial \mathcal{U}}{\partial \vec{e}}(\vec{r}) = \lim_{\lambda \rightarrow 0} \frac{\mathcal{U}(\vec{r} + \lambda \vec{e}) - \mathcal{U}(\vec{r})}{\lambda}.$$

Zaključena formula:

$$\frac{\partial \mathcal{U}}{\partial \vec{e}}(\vec{r}) = \vec{e} \text{ grad}\mathcal{U} = \vec{e}(\vec{\nabla}\mathcal{U}).$$

Odvod skalarnega polja \mathcal{U} v smeri danega vektorja \vec{a} :

$$\frac{\partial \mathcal{U}}{\partial \vec{a}} = \vec{a} \operatorname{grad} \mathcal{U} = \vec{a} (\vec{\nabla} \mathcal{U}).$$

Če \vec{a} ni nič, lahko pišemo $\vec{a} = |\vec{a}| \vec{e}$, kjer je \vec{e} enotski vektor; potem velja:

$$\frac{\partial \mathcal{U}}{\partial \vec{a}} = |\vec{a}| \vec{e} \operatorname{grad} \mathcal{U} = |\vec{a}| \frac{\partial \mathcal{U}}{\partial \vec{e}}.$$

Odvod danega vektorskega polja \vec{F} v smeri drugega danega vektorskega polja \vec{G} :

Naj bo $\vec{F} = (P, Q, R)$. Potem je

$$\frac{\partial \vec{F}}{\partial \vec{G}} = \left(\frac{\partial P}{\partial \vec{G}}, \frac{\partial Q}{\partial \vec{G}}, \frac{\partial R}{\partial \vec{G}} \right),$$

kjer so komponente vektorja na desni strani definirane kot odvodi komponent polja \vec{F} po polju \vec{G} . Torej lahko izrazimo:

$$\frac{\partial \vec{F}}{\partial \vec{G}} = (\vec{G} \operatorname{grad} P, \vec{G} \operatorname{grad} Q, \vec{G} \operatorname{grad} R) = (\vec{G} (\vec{\nabla} P), \vec{G} (\vec{\nabla} Q), \vec{G} (\vec{\nabla} R)).$$

Enak izraz bi dobili, če bi vzeli, da diferencialni operator

$$\vec{G} \vec{\nabla}$$

deluje zaporedno na komponente polja $\vec{F} = (P, Q, R)$. Zato lahko pišemo:

$$\frac{\partial \vec{F}}{\partial \vec{G}} = (\vec{G} \vec{\nabla}) \vec{F}.$$

Divergenca vektorskega polja $\vec{F} = (P, Q, R)$:

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \vec{\nabla} \cdot \vec{F}.$$

Rotor vektorskega polja $\vec{F} = (P, Q, R)$:

$$\operatorname{rot} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{\nabla} \times \vec{F}.$$

Opomba: v anglosaški literaturi pišejo namesto $\operatorname{rot} \vec{F}$ drugače: $\operatorname{curl} \vec{F}$.

Osnovni diferencialni operatorji prvega reda so linearni na linearnih kombinacijah s konstantnimi koeficienti α in β in enaki nič na konstantnih poljih \mathcal{C} oziroma \vec{C} :

$$\begin{aligned} \operatorname{grad}(\alpha\mathcal{U} + \beta\mathcal{V}) &= \alpha \operatorname{grad}\mathcal{U} + \beta \operatorname{grad}\mathcal{V} \\ (\vec{H} \vec{\nabla})(\alpha\vec{F} + \beta\vec{G}) &= \alpha(\vec{H} \vec{\nabla})\vec{F} + \beta(\vec{H} \vec{\nabla})\vec{G} \\ \operatorname{div}(\alpha\vec{F} + \beta\vec{G}) &= \alpha \operatorname{div} \vec{F} + \beta \operatorname{div} \vec{G} \\ \operatorname{rot}(\alpha\vec{F} + \beta\vec{G}) &= \alpha \operatorname{rot} \vec{F} + \beta \operatorname{rot} \vec{G} \end{aligned}$$

$$\begin{aligned} \operatorname{grad} \mathcal{C} &= \vec{0} \\ (\vec{G} \vec{\nabla})\vec{C} &= \vec{0} \\ \operatorname{div} \vec{C} &= 0 \\ \operatorname{rot} \vec{C} &= \vec{0} \end{aligned}$$

Če je \vec{a} konstanten, \vec{r} pa krajevni vektor, potem dobimo za osnovne operacije:

$$\operatorname{div} \vec{r} = 3, \operatorname{rot} \vec{r} = \vec{0}, \operatorname{grad}(\vec{a} \cdot \vec{r}) = \vec{a}, \operatorname{grad} |\vec{r}| = |\vec{r}|^{-1} \vec{r},$$

$$\operatorname{div}(\vec{a} \times \vec{r}) = 0, \operatorname{rot}(\vec{a} \times \vec{r}) = 2\vec{a}, (\vec{a} \vec{\nabla})\vec{r} = \vec{a}.$$

Druge pomembne enakosti

$$\operatorname{grad} \varphi(\mathcal{U}) = \varphi'(\mathcal{U}) \operatorname{grad} \mathcal{U}$$

$$\begin{aligned}\text{grad}(\mathcal{U}\mathcal{V}) &= \mathcal{U} \text{grad} \mathcal{V} + \mathcal{V} \text{grad} \mathcal{U} \\ \text{grad}(\vec{F} \vec{G}) &= (\vec{F} \vec{\nabla}) \vec{G} + (\vec{G} \vec{\nabla}) \vec{F} + \vec{F} \times \text{rot} \vec{G} + \vec{G} \times \text{rot} \vec{F}\end{aligned}$$

$$\begin{aligned}\text{div}(\mathcal{U} \vec{F}) &= \vec{F} \text{grad} \mathcal{U} + \mathcal{U} \text{div} \vec{F} \\ \text{div}(\vec{F} \times \vec{G}) &= \vec{G} \text{rot} \vec{F} - \vec{F} \text{rot} \vec{G}\end{aligned}$$

$$\begin{aligned}\text{rot}(\mathcal{U} \vec{F}) &= \text{grad} \mathcal{U} \times \vec{F} + \mathcal{U} \text{rot} \vec{F} \\ \text{rot}(\vec{F} \times \vec{G}) &= (\vec{G} \vec{\nabla}) \vec{F} - (\vec{F} \vec{\nabla}) \vec{G} + \vec{F} \text{div} \vec{G} - \vec{G} \text{div} \vec{F}\end{aligned}$$

Osnovni diferencialni operator drugega reda

Laplaceov operator, delta:

$$\Delta = \vec{\nabla} \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Za vektorsko polje $\vec{F} = (P, Q, R)$ je smiselno definirati:

$$\Delta \vec{F} = (\Delta P, \Delta Q, \Delta R).$$

Smisla nimajo:

$$\text{grad} \text{grad} \mathcal{U}, \text{grad} \text{rot} \vec{F}, \text{div} \text{div} \vec{F}, \text{rot} \text{div} \vec{F}.$$

Smisel pa imajo

$$\text{grad} \text{div} \vec{F}, \text{div} \text{grad} \mathcal{U}, \text{div} \text{rot} \vec{F}, \text{rot} \text{grad} \mathcal{U}, \text{rot} \text{rot} \vec{F},$$

in sicer veljajo enakosti:

$$\begin{aligned}\operatorname{rot} \operatorname{rot} \vec{F} &= \operatorname{grad} \operatorname{div} \vec{F} - \Delta \vec{F} \\ \operatorname{div} \operatorname{grad} \mathcal{U} &= \Delta \mathcal{U} \\ \operatorname{div} \operatorname{rot} \vec{F} &= 0 \\ \operatorname{rot} \operatorname{grad} \mathcal{U} &= \vec{0}\end{aligned}$$

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